Fréchet Distance Between Uncertain Trajectories: Computing Expected Value and Upper Bound*

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Abstract

A trajectory is a sequence of time-stamped locations. Measurement uncertainty is an important factor to consider when analysing trajectory data. We define an uncertain trajectory as a trajectory where at each time stamp the true location lies within an uncertainty region—a disk, a line segment, or a set of points. In this paper we consider discrete and continuous Fréchet distance between uncertain trajectories.

We show that finding the largest possible discrete or continuous Fréchet distance among all possible realisations of two uncertain trajectories is NP-hard under all the uncertainty models we consider. Furthermore, computing the expected discrete or continuous Fréchet distance is #P-hard when the uncertainty regions are modelled as point sets or line segments. We also study the setting with time bands, where we restrict temporal alignment of the two trajectories, and give polynomial-time algorithms for largest possible and expected discrete and continuous Fréchet distance for uncertainty regions modelled as point sets.

1 Introduction

Trajectory data is ubiquitous. Whether tracking animals or dissecting a football game, we need to deal with automated analysis of measured trajectories. However, most existing approaches do not take into account the inherent uncertainty that arises due to the measurement procedure. In some settings this uncertainty is small on the scale of the analysis; in other settings, however, meaningful results can only be obtained when dealing with such uncertainty explicitly. In this paper, we aim to do that for a variety of uncertainty models when computing Fréchet distance and discrete Fréchet distance.

There are many results on trajectory analysis: on simplification of trajectories [1, 14, 21, 22, 29]; on trajectory segmentation [3, 4, 6]; on clustering trajectories [8, 17]. There are also many approaches to trajectory similarity [13, 25, 31], including (discrete) Fréchet distance [5, 16, 20] and variants [15]. There is some work tackling uncertainty in computational geometry [12, 24, 26, 27], including problems on moving points [10, 18].

Some authors suggest computing restricted versions of Fréchet distance and other distance metrics using time bands [7, 23, 30], restricting the alignment of trajectories. This is mostly useful when the trajectories are regularly sampled and are expected to be aligned in time, so we can use some fixed-size band on indices of the trajectory points as proxy for timestamps.

* This abstract presents partial results from joint work with Chenglin Fan and Benjamin Raichel [9].

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Table 1 Summary of hardness results for the decision problems in this paper.

		indecisive	imprecise	
			disks	line segments
discrete FD	UB	NP-complete	NP-complete	NP-complete
	Exp	#P-hard	—	#P-hard
FD	UB	NP-complete	NP-complete	NP-complete
	Exp	#P-hard	—	—

There is some work on similarity of uncertain trajectories. Buchin and Sijben [11] study computing discrete Fréchet distance on uncertain trajectories with points defined by probability distributions. Ahn et al. [2] model each uncertain point by a disk, and the real location of a point may be any point in the disk. They compute the lowest possible discrete Fréchet distance using a dynamic programming approach. They also stipulate that finding largest possible Fréchet distance is hard; it is confirmed by Fan and Zhu [19] for the case of thin rectangles as imprecision model and is further explored in this paper.

We focus on Fréchet distance and discrete Fréchet distance. We make a distinction between *indecisive* points and *imprecise* points for location uncertainty, as explained in Section 1.1. We only model measurement uncertainty, so we assume linear motion on a straight line segment between two consecutive measurements. We consider *upper bound* Fréchet distance and *expected* Fréchet distance between trajectories, which correspond to the largest possible and expected Fréchet distance over every possible combination of real locations of the trajectory. Our contributions are:

1. NP-hardness and #P-hardness results.[1] We show NP-hardness for the upper bound on (discrete) Fréchet distance using simpler uncertainty regions and a simpler construction than Fan and Zhu [19]. We show #P-hardness for the expected value of (discrete) Fréchet distance in several settings. See Table 1 for details.
2. Algorithms for discrete and continuous Fréchet distance with Sakoe–Chiba time bands. Previous results suggest that there is little room for positive algorithmic results. If the trajectories are regularly sampled, or can be resampled appropriately at will, and are expected to align in time, we can restrict the computation to a fixed-width time window on indices of trajectory points, as explained in Section 3. We give algorithms to find, given *indecisive* trajectories, the upper bound and expected (discrete) Fréchet distance when constrained to Sakoe–Chiba bands of fixed width [30].

The results of this abstract are discussed further in the master thesis of A. Popov [28] and in joint work with C. Fan and B. Raichel [9]. In the latter paper, we additionally investigate the lower bound (continuous) Fréchet distance.

1.1 Notation

We denote a *polygonal curve* of length n on n points in d dimensions as $P = \langle p_1, p_2, \dots, p_n \rangle$. A *trajectory* is a polygonal curve with timestamps associated to each point of the curve. Whenever timestamps are not relevant, we use the terms interchangeably. We denote a *subtrajectory* from point i to j of curve P as $P[i : j]$.

[1] Hardness class #P is a class of counting problems related to NP. For example, SAT ('Is there a satisfying assignment to a boolean formula?') is an NP-complete problem, whereas #SAT ('How many satisfying assignments to a boolean formula are there?') is a #P-complete problem.

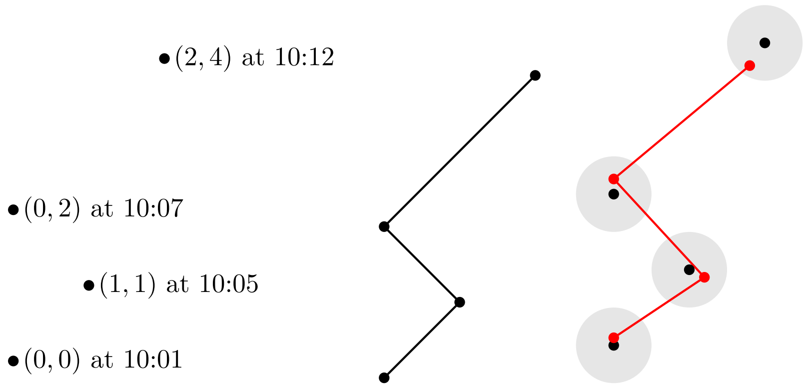


Figure 1 Left: Trajectory data. Centre: Polygonal curve on the data. Right: Imprecise trajectory with disks as imprecision regions and the real trajectory.

An *uncertain* point is commonly represented as a compact region $H \subset \mathbb{R}^d$. A *realisation* of such a point h is one of the points from the region H. An *indecisive* point is a special case of an uncertain point: it is a set of points $H = \{h_1, \dots, h_k\}$. Similarly, an *imprecise* point is a compact connected region $H \subset \mathbb{R}^d$. We use disks or line segments as such regions. Note that a precise point is a special case of an indecisive point and an imprecise point.

Consider a sequence of uncertain points $\mathcal{H} = \langle H_1, \dots, H_n \rangle$, referred to as an *uncertain trajectory*. A *realisation* $P \Subset \mathcal{H}$ of an uncertain trajectory is a polygonal curve $P = \langle p_1, \dots, p_n \rangle$, where each p_i is a realisation of the corresponding uncertain point H_i. The concept of uncertain trajectories is illustrated in Figure 1.

Extending the notation to uncertain trajectories $\mathcal{H}$ and $\mathcal{V}$, we define the upper bound on the (discrete) Fréchet distance under different possible realisations:

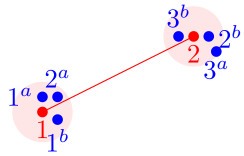
$$d_{\mathrm{dF}}^{\max}(\mathcal{H}, \mathcal{V}) = \max_{A \Subset \mathcal{H}, B \Subset \mathcal{V}} d_{\mathrm{dF}}(A, B)\,, \qquad d_{\mathrm{F}}^{\max}(\mathcal{H}, \mathcal{V}) = \max_{A \Subset \mathcal{H}, B \Subset \mathcal{V}} d_{\mathrm{F}}(A, B)\,.$$

We define *expected Fréchet distance* $d_{\mathrm{dF}}^{\mathbb{E}}$ and $d_{\mathrm{F}}^{\mathbb{E}}$ as the expected value of the Fréchet distance if the realisations are picked uniformly at random, independently for each trajectory point.

2 Hardness Results

We do not discuss the construction; see the master thesis for full proofs [28]. It is possible to provide a reduction from CNF-SAT to the decision problem for finding $d_{\mathrm{dF}}^{\max}$ and $d_{\mathrm{F}}^{\max}$ under different uncertainty models, establishing their NP-hardness. Furthermore, it is possible to provide reductions from the counting version of CNF-SAT to the decision problem for finding $d_{\mathrm{dF}}^{\mathbb{E}}$ and $d_{\mathrm{F}}^{\mathbb{E}}$ in some settings, establishing their #P-hardness. The construction has two trajectories, one precise and one uncertain; every realisation of the uncertain trajectory corresponds to a variable assignment in the CNF-SAT formula. We get two possible values of Fréchet distance for each realisation and can distinguish satisfying assignments. Then $d_{\mathrm{dF}}^{\max}$ tells us if the formula is satisfiable, and $d_{\mathrm{dF}}^{\mathbb{E}}$ gives us the count of satisfying assignments.

The proofs using our construction extend to other compact uncertainty regions of the same shape and size for the discrete Fréchet distance; the extension for the continuous Fréchet distance seems possible, but is less obvious. The expected case is a lot more difficult due to the complicated integral evaluations, so even for disks the results seem difficult to obtain. The list of settings we consider is shown in Table 1.



F F	F T	T T
T T	T F	F F

F
T

→

F	T
T	F

→

T
F

Figure 2 Left: An indecisive and a precise trajectory. Middle: Distance matrix. 'T T' in the bottom left cell means $\|1-1^a\| \leq \varepsilon$ and $\|1-1^b\| \leq \varepsilon$. Right: Computing reachability matrix, column by column. Note the two reachability vectors for the second column.

3 Algorithms with Time Bands

Here we use the *Sakoe–Chiba band,* which restricts aligning point k on one trajectory to points $k \pm w$ on the other trajectory, for all k and some fixed w [30]. In some settings the point indices act as proxy for timestamps, and the trajectories are expected to be aligned in time, so this restriction is reasonable. We develop polynomial-time algorithms for the restricted hard problems of the previous section on *indecisive* points.

3.1 Upper Bound Discrete Fréchet Distance

First of all, let us discuss a simple setting. Suppose we are given a trajectory $V = \langle q_1, \dots, q_n \rangle$ of n precise points and $\mathcal{H} = \langle P_1, \dots, P_n \rangle$ of n indecisive points, each of them having ℓ options, so for all $i \in \{1, \dots, n\}$ we have $P_i = \{p_i^1, \dots, p_i^\ell\}$. We would like to answer the following decision problem: *'If we restrict the couplings to a Sakoe–Chiba band of width w, is it true that $\mathrm{d}_{\mathrm{dF}}^{\max}(\mathcal{H}, V) \leq \varepsilon$ for some given threshold $\varepsilon > 0$?'* We want to solve the decision problem for the upper bound discrete Fréchet distance between a precise and an indecisive trajectory.

In a fully precise setting the discrete Fréchet distance can be computed using dynamic programming [16]. We create a table where the rows correspond to the vertices of one trajectory, say V, and columns correspond to the vertices of the other trajectory, say H. Each table entry (i, j) then contains a True or False value indicating if there is a coupling between $V[1:j]$ and $H[1:i]$ with maximum distance at most ε. We use a similar approach.

Suppose we position $\mathcal{H}$ to go horizontally along the table, and V to go vertically. Consider an arbitrary column in the table and suppose that we fix the realisation of a part of $\mathcal{H}$ up to the previous column. Then we can simply consider the new column ℓ times, each time picking a different realisation for the new point on $\mathcal{H}$, and compute the resulting reachability. As we do this for the entire column at once, we can ensure consistency of our choice of realisation of $\mathcal{H}$. This procedure will give us a set of binary reachability vectors for the new column, each vector corresponding to a realisation of a prefix of $\mathcal{H}$. The *reachability vector* is a boolean vector that, for the cell (i, j) of the table, states whether for a particular realisation A of $\mathcal{H}[1:i]$ the discrete Fréchet distance between A and $V[1:j]$ is below some threshold ε.

An important observation is that we do not need to distinguish between the realisations of trajectory prefixes that give the same reachability vector: once we start filling out the next column, all we care about is the existence of some realisation leading to that particular reachability vector. So, we can keep a *set* of binary vectors of reachability in the column.

This procedure was suggested for a specific realisation of a prefix of $\mathcal{H}$. However, we can also repeat this for each previous reachability vector, only keeping the unique results. As all the realisation choices happen along $\mathcal{H}$, by treating the table column-by-column we ensure that we do not have issues with inconsistent choices. Therefore, repeating this procedure n times, we will fill out the last column of the table. At that point, if any vector has False in the top right cell, then there is some realisation $A \Subset \mathcal{H}$ such that $d_{\mathrm{dF}}(A, V) > \varepsilon$, and hence $d_{\mathrm{dF}}^{\max}(\mathcal{H}, V) > \varepsilon$; otherwise, $d_{\mathrm{dF}}^{\max}(\mathcal{H}, V) \leq \varepsilon$, as there are no 'bad' realisations.

In more detail, we use two tables, distance matrix D and reachability matrix R. First of all, we initialise the distance matrix D and the reachability of the first column for all possible locations of H_1. Then we fill out R column-by-column. We take the reachability of the previous column and note that any cell can be reached either with the horizontal step or with the diagonal step. We need to consider various extensions of the trajectory $\mathcal{H}$ with one of the ℓ realisations of the current point: the distance matrix should allow the specific coupling. Furthermore, assume we find that a certain cell is reachable; if allowed by the distance matrix, we can then go upwards, marking cells above the current cell reachable, even if they are not directly reachable with a horizontal or diagonal step. Then we just remember the newly computed vector; we make sure to only remember distinct vectors.

We check if there is a realisation that yields FALSE in the last cell; then this realisation is chosen by the upper bound, yielding FALSE. The computation is illustrated in Figure 2.

We can extend this approach to the setting where both trajectories are indecisive, so instead of V we have $\mathcal{V} = \langle V_1, \dots, V_n \rangle$, with, for each $j \in \{1, \dots, n\}$, $V_j = \{q_j^1, \dots, q_j^\ell\}$.

Suppose we pick a realisation for trajectory $\mathcal{V}$. Then we can apply the algorithm we just described. We cannot run it separately for every realisation of $\mathcal{V}$; instead, note that the part of the realisation that matters for column i is the points from $i - w$ to $i + w$, since any previous or further points are outside the time band. We can fix these $2w + 1$ points and compute the column as before; we do so for each possible combination on these $2w + 1$ points.

▶ **Theorem 1.** *Suppose we are given two indecisive trajectories of length n with ℓ options per indecisive point. Then we can compute the upper bound discrete Fréchet distance restricted to a Sakoe–Chiba band of width w in time $\Theta(4^w n \sqrt{w} \ell^{2w})$.*

3.2 Expected Discrete Fréchet Distance

To compute the expected discrete Fréchet distance with time bands, we need two observations:

1. For any two precise trajectories, there is a single threshold ε where the answer to the decision problem changes from TRUE to FALSE—a *critical value*. That threshold corresponds to the distance between some two points on the trajectories.
2. We can modify our algorithm to store associated counts with each reachability vector, obtaining the fraction of realisations that yield the answer TRUE for a given threshold ε.

So, we can execute our algorithm for each of the critical values and obtain the cumulative distribution function $\mathbb{P}(d_{\mathrm{dF}}(A, B) > \varepsilon)$ for $A, B \Subset \mathcal{H}, \mathcal{V}$ following the uniform distribution. Since the cumulative distribution function is a step function, we can compute $d_{\mathrm{dF}}^{\mathbb{E}}$.

▶ **Theorem 2.** *Suppose we are given two indecisive trajectories $\mathcal{H}$ and $\mathcal{V}$ of length n with ℓ options per indecisive point. Then we can compute the expected discrete Fréchet distance when constrained to a Sakoe–Chiba band of width w in time $\Theta(4^w n^2 w^2 \ell^{2w})$ in the worst case.*

3.3 Continuous Fréchet Distance

We can adapt our time band algorithms to handle the continuous Fréchet distance. Instead of the boolean reachability vectors, we use columns of *free space* cells, introduced by Alt and Godau [5, 20], as illustrated in Figure 3. We store the reachability intervals on cell borders.

The specifics of handling intervals are very technical and can be found in the master thesis [28]. The number of possible intervals is bounded; this way we get an algorithm that runs in time polynomial in n. An extension to find the expected value is also possible.

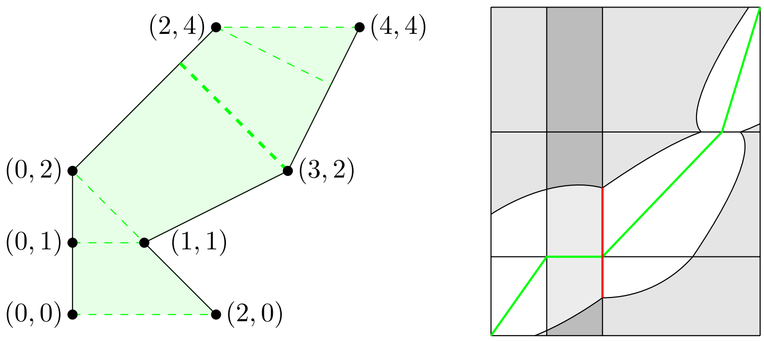


Figure 3 Left: Visualisation of Fréchet distance on precise trajectories. Right: Corresponding free-space diagram. The highlighted intervals are propagated per column in the uncertain case. The monotone path corresponds to the alignment depicted on the left.

▶ **Theorem 3.** *Suppose we are given two indecisive trajectories of length n with ℓ options per indecisive point. Then we can compute the upper bound Fréchet distance and the expected Fréchet distance restricted to a Sakoe–Chiba band of fixed width w in time polynomial in n.*

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